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Local Densities of Alternating Forms*

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We give an explicit formula for local densities of integral representations of nondegenerate alternating matrices with entries in the ring of $\mathfrak{p}$-adic integers in terms of elementary divisors. The proof of the formula is based on the local functional equation satisfied by the zeta function on the space of alternating forms and some properties of spherical functions. © 1989 Academic Press, Inc.

0. INTRODUCTION

In the present paper we apply the theory of zeta functions on the space of alternating matrices over a $\mathfrak{p}$-adic number field to calculation of local densities of integral representations of alternating forms.

Let k be a $\mathfrak{p}$-adic number field, $\mathfrak{o}$ the ring of integers of k, and $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$. Put $q = N(\mathfrak{p})$. Let m and n be positive integers with $m \geqslant n$. For nondegenerate alternating matrices A and B of sizes $2m$ and $2n$, respectively, with entries in $\mathfrak{o}$, we define the local density of integral representations of B by A by the limit

$$\mu(B, A) = \lim_{l \to \infty} (q^{-n(4m-2n+1)})^l \times \#\{T \in M(2m, 2n; \mathfrak{o}/\mathfrak{p}^l);\ {}^tTAT \equiv B \pmod{\mathfrak{p}^l}\}.$$

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The goal of this paper is to give an explicit formula for $\mu(B, A)$ in terms of the elementary divisors of A and B.

Put

$$\Lambda_n^+ = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n; \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$$

and fix a prime element π of k. Then by the theory of elementary divisors, we know that A and B are equivalent to matrices of the forms

$$\pi^\xi = \begin{pmatrix} 0 & \pi^{\xi_1} \\ -\pi^{\xi_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\xi_m} \\ -\pi^{\xi_m} & 0 \end{pmatrix} \qquad (\xi \in \Lambda_m^+),$$

$$\pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \qquad (\lambda \in \Lambda_n^+),$$

respectively.

For any $\mu \in \Lambda_n^+$, we put

$$|\mu| = \sum_{i=1}^n \mu_i, \qquad n(\mu) = \sum_{i=1}^n (i-1)\mu_i,$$

$$\mu_i' = \#\{j; \mu_j \geqslant i\} \ (i \geqslant 1)$$

and for $n \geqslant r \geqslant 0$,

$$\begin{bmatrix} n \\ r \end{bmatrix}(t) = \prod_{i=1}^n (1-t^i) \Big/ \left\{ \prod_{i=1}^r (1-t^i) \cdot \prod_{i=1}^{n-r} (1-t^i) \right\}.$$

Now the main result of this paper can be formulated as follows:

THEOREM. *We have*

$$\begin{aligned} \mu(B, A) &= \mu(\pi^\lambda, \pi^\xi) \\ &= \sum_{\substack{\mu \in \Lambda_n^+ \\ \mu_i' \leqslant \tilde{\lambda}_i'}} (-1)^{|\mu|} q^{2(n-m-1)|\mu| - 2n(\mu) + 2\langle \xi', \mu' \rangle} \\ &\quad \times \prod_{\substack{i \geqslant 1 \\ \mu_i' \neq \mu_{i+1}'}} \left\{ \sum_{l=\mu_{i+1}'}^{\min\{\tilde{\lambda}_{i+1}', \mu_i'\}} (-q)^{l(2\tilde{\lambda}_{i+1}' - l)} \right. \\ &\quad \left. \times \begin{bmatrix} \tilde{\lambda}_{i+1}' - \mu_{i+1}' \\ \tilde{\lambda}_{i+1}' - l \end{bmatrix} (q^{-2}) \begin{bmatrix} \tilde{\lambda}_i' - l \\ \tilde{\lambda}_i' - \mu_i' \end{bmatrix} (q^{-2}) \right\}, \end{aligned}$$

where $\tilde{\lambda} = (\lambda_1 + 1, ..., \lambda_n + 1)$ *and* $\langle \xi', \mu' \rangle = \sum_{i \geqslant 1} \xi_i' \mu_i'$.

In a previous paper [HS], we presented a formula for $\mu(\pi^\lambda, \pi^\xi)$ [HS, Theorem 8], which is less explicit than the present one. The derivation of the formula in [HS] is based on the fact that the spherical function on the space of alternating forms can be viewed as a generating function of local densities. Here we employ another approach to local densities, though spherical functions still play an important role.

One of the standard methods of calculating local densities is to use Gaussian sums. In the present case we consider the Gaussian

$$\mu(\pi^\lambda, \pi^\xi) = q^{-4lmn} \sum_T \sum_Y \chi(\operatorname{tr}(Y({}^tT\pi^\xi T - \pi^\lambda))/2),$$

where l is a sufficiently large integer, χ is an additive character of $\mathfrak{o}/\mathfrak{p}^l$ which is nontrivial on $\mathfrak{p}^{l-1}/\mathfrak{p}^l$, T runs through $M(2m, 2n; \mathfrak{o}/\mathfrak{p}^l)$, and Y runs through the set of all alternating matrices of size $2n$ with entries in $\mathfrak{o}/\mathfrak{p}^l$. As we shall see later in Section 1, the hardest part of the calculation of the Gaussian sum is computing the Fourier transform $\hat{c}_\lambda$ of the characteristic function c_λ of the set of integral alternating matrices equivalent to π^λ over $\mathfrak{o}$.

Our tool in calculating $\hat{c}_\lambda$ is the zeta function

$$Z(\hat{c}_\lambda; s) = \int \prod_{i=1}^{n} |\mathrm{Pf}_i(x)|_{\mathfrak{p}}^{s_i} \hat{c}_\lambda(x) \left| \frac{dx}{\mathrm{Pf}_n(x)^{2n-1}} \right|_{\mathfrak{p}},$$

where $\mathrm{Pf}_i(x)$ is the Pfaffian of the upper left $2i$ by $2i$ block of x and the integral is taken over all alternating matrices of size $2n$ with entries in k satisfying $\mathrm{Pf}_i(x) \neq 0$ for all $i = 1, 2, ..., n$. The theory of spherical functions on the space of alternating forms (cf. [HS]) shows that the function $\hat{c}_\lambda$ can be determined completely by the zeta function $Z(\hat{c}_\lambda; s)$ (Section 2, Theorem 2 (ii)) and hence one can extract all the necessary information on $\hat{c}_\lambda$ from $Z(\hat{c}_\lambda; s)$ (at least in principle), once he obtains its appropriate explicit formula. Another important fact is that $Z(\hat{c}_\lambda; s)$ is a typical example of zeta functions associated with prehomogeneous vector spaces defined over a $\mathfrak{p}$-adic number field (cf. [I], [S]) and satisfies a functional equation connecting it to $Z(c_\lambda; s)$ (Section 2, Theorem 1). This fact together with the theory of spherical functions gives us a method of deriving an explicit formula for $Z(\hat{c}_\lambda; s)$ suitable for our purpose. After establishing in Section 2 some fundamental properties of zeta functions on the space of alternating matrices, we compute $Z(\hat{c}_\lambda; s)$ and prove the main theorem in Section 3.

Local densities have been considered mainly for quadratic forms and precise information on local densities will certainly have several interesting applications to the arithmetic of quadratic forms (see, e.g., [Si], [K]). Therefore it is natural to ask whether a similar method can also be applied to quadratic forms. The results in Section 1 on Gaussian sums have a counterpart for quadratic forms (cf. [H2]). The functional equation of zeta functions has been proved also for the space of symmetric matrices in [S, Section 3, Theorem 3.2]. The main obstacle to extending our results to quadratic forms is the theory of spherical functions. The present state of the theory of spherical functions on the space of symmetric matrices is still unsatisfactory, though several important properties including functional equations of spherical functions and the injectivity of the spherical Fourier transform have been obtained in [H1].

1. Gaussian Sum

Let k be a $\mathfrak{p}$-adic number field, $\mathfrak{o}$ the ring of integers of k, and $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$. Fix a prime element π of k: $\mathfrak{p} = (\pi)$. We put $q = N(\mathfrak{p})$.

For $R = k$, $\mathfrak{o}$, or $\mathfrak{o}/\mathfrak{p}^l$ and a positive integer n, denote by $V_n(R)$ the set of alternating matrices of size $2n$ with entries in R. Then the group $G_n(R) = GL(2n, R)$ acts on $V_n(R)$ via

$$x \mapsto {}^t gxg \qquad (x \in V_n(R),\ g \in G_n(R)).$$

For $R = k$ or $\mathfrak{o}$, we let $X_n(R)$ be the set of nondegenerate alternating matrices in $V_n(R)$. The set $X_n(R)$ is stable under the action of $G_n(R)$. For simplicity we put $K_n = G_n(\mathfrak{o})$.

Put

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n;\ \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

For a $\lambda \in \Lambda_n^+$, put

$$\pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \ (\in X_n(\mathfrak{o})).$$

Then a complete set of representatives of $K_n \backslash X_n(\mathfrak{o})$ is given by $\{\pi^\lambda;\ \lambda \in \Lambda_n^+\}$.

Let m and n be positive integers with $m \geqslant n$. For $A \in X_m(\mathfrak{o})$ and $B \in X_n(\mathfrak{o})$, denote by $N_l(B, A)$ the number of solutions T in $M(2m, 2n; \mathfrak{o}/\mathfrak{p}^l)$ of the congruence

$${}^t TAT \equiv B \qquad (\mathrm{mod}\ \mathfrak{p}^l).$$

Then the local density of integral representations of B by A is defined by the limit

$$\mu(B, A) = \lim_{l \to \infty} (q^{-n(4m-2n+1)})^l N_l(B, A).$$

The local density $\mu(B, A)$ depends only on the K_n-orbit containing B and the K_m-orbit containing A. Therefore our problem is to find an explicit formula expressing $\mu(\pi^\lambda, \pi^\xi)$ ($\lambda \in \Lambda_n^+$, $\xi \in \Lambda_m^+$) in terms of λ and ξ.

For $Y = (y_{ij})$, $Y^* = (y_{ij}^*) \in V_n(\mathfrak{o})$ or $V_n(\mathfrak{o}/\mathfrak{p}^l)$, we put

$$\langle Y, Y^* \rangle = \sum_{1 \leqslant i < j \leqslant n} y_{ij} y_{ij}^* = \operatorname{tr}(Y \cdot {}^tY^*)/2.$$

Let $\chi \colon \mathfrak{o}/\mathfrak{p}^l \to \mathbb{C}^\times$ be an additive character which is nontrivial on $\mathfrak{p}^{l-1}/\mathfrak{p}^l$. Then, by the orthogonality relation of characters of $\mathfrak{o}/\mathfrak{p}^l$, we have

$$N_l(B, A) = (q^{-n(2n-1)})^l \sum_T \sum_Y \chi(\langle Y, {}^tTAT - B \rangle),$$

where T (resp. Y) runs through $M(2m, 2n; \mathfrak{o}/\mathfrak{p}^l)$ (resp. $V_n(\mathfrak{o}/\mathfrak{p}^l)$). In the following we always assume that l is large enough to satisfy the condition that

$$\text{the } K_n\text{-orbit containing } B \text{ decomposes into cosets modulo } \mathfrak{p}^l. \tag{1.1}$$

Then we have

$$(q^{-n(4m-2n+1)})^l N_l(B, A) = \operatorname{vol}(T(B, A))/\operatorname{vol}(K_n \cdot B),$$

where $K_n \cdot B$ is the K_n-orbit containing B,

$$T(B, A) = \{T \in M(2m, 2n; \mathfrak{o}); {}^tTAT \in K_n \cdot B\},$$

and the volumes of $K_n \cdot B$ and $T(B, A)$ are measured by the normalized Haar measures on $V_n(k)$ and $M(2m, 2n; k)$, respectively. Therefore under Condition (1.1) $(q^{-n(4m-2n+1)})^l N_l(B, A)$ is independent of l and the local density $\mu(B, A)$ is given by the formula

$$\mu(B, A) = q^{-4lmn} \sum_T \sum_Y \chi(\langle Y, {}^tTAT - B \rangle). \tag{1.2}$$

Take an additive character $\psi \colon k \to \mathbb{C}^\times$ of k such that ψ is trivial on $\mathfrak{p}^l$ and the character of $\mathfrak{o}/\mathfrak{p}^l$ obtained from ψ coincides with χ. Denote by $\mathscr{S}(V_n(k))$ the space of Schwartz–Bruhat functions on $V_n(k)$. Using the character ψ, we define a Fourier transform $\hat{f}$ of $f \in \mathscr{S}(V_n(k))$ by the formula

$$\hat{f}(Y^*) = \int_{V_n(k)} f(Y)\, \psi(\langle Y, Y^* \rangle)\, dY,$$

where dY is the Haar measure on $V_n(k)$ normalized by $\int_{V_n(\mathfrak{o})} dY = 1$. Note that a complete set of representatives of $G_n(\mathfrak{o}/\mathfrak{p}^l)\backslash V_n(\mathfrak{o}/\mathfrak{p}^l)$ is given by $\{\pi^\lambda (\text{mod } \mathfrak{p}^l); \lambda \in \Lambda_{n,l}^+\}$, where

$$\Lambda_{n,l}^+ = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n; l \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

LEMMA 1. *For $\lambda \in \Lambda_n^+$ and $\xi \in \Lambda_m^+$,*

$$\mu(\pi^\lambda, \pi^\xi) = q^{-ln(4m-2n+1)} \sum_{\mu \in \Lambda_{n,l}^+} \frac{N_l^{pr}(\pi^\lambda, \pi^\lambda)}{N_l^{pr}(\pi^\mu, \pi^\mu)} \cdot S_l(\pi^\xi, \pi^\mu)\, \hat{c}_\lambda(\pi^\mu),$$

where

$$N_l^{pr}(\pi^\lambda, \pi^\lambda) = \#\{g \in G_n(\mathfrak{o}/\mathfrak{p}^l); {}^t g\pi^\lambda g \equiv \pi^\lambda \ (\text{mod } \mathfrak{p}^l)\},$$

$$S_l(\pi^\xi, \pi^\mu) = \sum_T \chi(\langle \pi^\mu, {}^t T\pi^\xi T\rangle) \qquad (T \in M(2m, 2n; \mathfrak{o}/\mathfrak{p}^l)),$$

and c_λ $(\in \mathcal{S}(V_n(k)))$ is the characteristic function of the K_n-orbit containing π^λ.

Proof. By exchanging the order of the summation in the right hand side of (1.2), we obtain

$$\mu(\pi^\lambda, \pi^\xi) = q^{-4lmn} \sum_Y \chi(-\langle Y, \pi^\lambda\rangle)\, S_l(\pi^\xi, Y).$$

Since $S_l(\pi^\xi, Y)$ depends only on the $G_n(\mathfrak{o}/\mathfrak{p}^l)$-equivalence class of Y, we get

$$\mu(\pi^\lambda, \pi^\xi) = q^{-4lmn} \sum_{\mu \in \Lambda_{n,l}^+} \left\{ \sum_{Y \sim \pi^\mu} \chi(-\langle Y, \pi^\lambda\rangle) \right\} \cdot S_l(\pi^\xi, \pi^\mu),$$

where the summation with respect to Y is taken over the $G_n(\mathfrak{o}/\mathfrak{p}^l)$-orbit containing π^μ mod $\mathfrak{p}^l$ in $V_n(\mathfrak{o}/\mathfrak{p}^l)$. It is clear that $\langle {}^t gYg, Y^*\rangle = \langle Y, {}^t gY^*g\rangle$. Therefore

$$\begin{aligned} \sum_{Y \sim \pi^\mu} \chi(-\langle Y, \pi^\lambda\rangle) &= N_l^{pr}(\pi^\mu, \pi^\mu)^{-1} \sum_{g \in G_n(\mathfrak{o}/\mathfrak{p}^l)} \chi(-\langle {}^t g\pi^\mu g, \pi^\lambda\rangle) \\ &= \{N_l^{pr}(\pi^\lambda, \pi^\lambda)/N_l^{pr}(\pi^\mu, \pi^\mu)\} \cdot \sum_{Y \sim \pi^\lambda} \chi(-\langle \pi^\mu, Y\rangle). \end{aligned}$$

Now the lemma follows from the formula

$$\sum_{Y \sim \pi^\lambda} \chi(-\langle \pi^\mu, Y\rangle) = q^{ln(2n-1)} \hat{c}_\lambda(\pi^\mu),$$

which is an immediate consequence of Condition (1.1).

For a non-negative integer m, put

$$w_m(t)=\prod_{j=1}^{m}(1-t^j), \tag{1.3}$$

and for $\lambda\in\Lambda_n^+$, define

$$w_\lambda^{(n)}(t)=\prod_{j=0}^{\infty}w_{m_j(\lambda)}(t),$$

where $m_j(\lambda)$ is the multiplicity of j in λ, namely,

$$m_j(\lambda)=\#\{i;\lambda_i=j,\ 1\leqslant i\leqslant n\}.$$

We further put

$$n(\lambda)=\sum_{i=1}^{n}(i-1)\lambda_i \qquad \text{and} \qquad |\lambda|=\sum_{i=1}^{n}\lambda_i.$$

LEMMA 2. *For* $\lambda\in\Lambda_{n,l}^+$,

$$N_l^{\mathrm{pr}}(\pi^\lambda,\pi^\lambda)=q^{n(2n+1)l+4n(\lambda)+|\lambda|}\cdot\frac{w_{2m_l(\lambda)}(q^{-1})}{w_{m_l(\lambda)}(q^{-2})}\cdot w_\lambda^{(n)}(q^{-2}).$$

Proof. Suppose that $\lambda_{n-r}>\lambda_{n-r+1}$ $(1\leqslant r\leqslant n-1)$ and put $\nu=(\lambda_1,...,\lambda_{n-r})$ and $\mu=(\lambda_{n-r+1},...,\lambda_n)$. Let us prove the recursion formula

$$N_l^{\mathrm{pr}}(\pi^\lambda,\pi^\lambda)=q^{4(n-r)(|\mu|+nl)}N_l^{\mathrm{pr}}(\pi^\mu,\pi^\mu)\,N_l^{\mathrm{pr}}(\pi^\nu,\pi^\nu). \tag{1.4}$$

First note that any primitive element v in $M(2n,2r;\mathfrak{o}/\mathfrak{p}^l)$ satisfying the congruence ${}^tv\pi^\lambda v\equiv\pi^\mu \pmod{\mathfrak{p}^l}$ can be extended to an element U in $G_n(\mathfrak{o}/\mathfrak{p}^l)$ satisfying ${}^tU\pi^\lambda U\equiv\pi^\lambda \pmod{\mathfrak{p}^l}$. Two such extensions U and V are related as follows:

$$U^{-1}V=\left(\begin{array}{c|c}A&0\\\hline B&I_{2r}\end{array}\right),\qquad A\in G_{n-r}(\mathfrak{o}/\mathfrak{p}^l),\ {}^tA\pi^\nu A\equiv\pi^\nu \pmod{\mathfrak{p}^l},$$

$$B\in M(2r,2(n-r);\mathfrak{o}/\mathfrak{p}^l),\ \pi^\mu B\equiv 0 \pmod{\mathfrak{p}^l}.$$

Hence

$$N_l^{\mathrm{pr}}(\pi^\lambda,\pi^\lambda)=q^{4(n-r)|\mu|}N_l^{\mathrm{pr}}(\pi^\mu,\pi^\lambda)\,N_l^{\mathrm{pr}}(\pi^\nu,\pi^\nu),$$

where $N_l^{\mathrm{pr}}(\pi^\mu,\pi^\lambda)$ is the number of primitive v satisfying ${}^tv\pi^\lambda v\equiv\pi^\mu$

(mod $\mathfrak{p}^l$). We write $v = \binom{v_1}{v_2}$ with $v_1 \in M(2(n-r), 2r; \mathfrak{o}/\mathfrak{p}^l)$ and $v_2 \in M(2r; \mathfrak{o}/\mathfrak{p}^l)$. Then the congruence satisfied by v is equivalent to

$$ {}^t v_1 \pi^\nu v_1 + {}^t v_2 \pi^\mu v_2 \equiv \pi^\mu \qquad (\text{mod } \mathfrak{p}^l). $$

Since $\lambda_{n-r} > \lambda_{n-r+1}$ and v is primitive, v_2 must be primitive. Moreover for any fixed v_1 there exist primitive v_2 satisfying the congruence above. Any two such v_2 and v_2' satisfy the relation

$$ {}^t(v_2' v_2^{-1})\, \pi^\mu (v_2' v_2^{-1}) \equiv \pi^\mu \qquad (\text{mod } \mathfrak{p}^l). $$

Therefore we obtain

$$ N_l^{\mathrm{pr}}(\pi^\mu, \pi^\lambda) = q^{4r(n-r)l} N_l^{\mathrm{pr}}(\pi^\mu, \pi^\mu). $$

Thus we have proved Formula (1.4).

Now we write $\lambda = (\mu_1^{t_1} \cdots \mu_s^{t_s})$ with $\mu_1 > \cdots > \mu_s \geqslant 0$, $t_i := m_{\mu_i}(\lambda) \geqslant 1$, and $t_1 + \cdots + t_s = n$. Put $\lambda^{(i)} = (\mu_i, ..., \mu_i) \in \Lambda_{t_i}^+$ $(1 \leqslant i \leqslant s)$. Then by (1.4)

$$ N_l^{\mathrm{pr}}(\pi^\lambda, \pi^\lambda) = q^{4\sum_{i=2}^{s} (\mu_i t_i + t_i l) \sum_{j=1}^{i-1} t_j} \prod_{i=1}^{r} N_l^{\mathrm{pr}}(\pi^{\lambda^{(i)}}, \pi^{\lambda^{(i)}}). $$

The lemma follows from this identity and the formula

$$ N_l^{\mathrm{pr}}(\pi^{\lambda^{(i)}}, \pi^{\lambda^{(i)}}) = \begin{cases} q^{t_i(2t_i-1)\mu_i + t_i(2t_i+1)l} w_{t_i}(q^{-2}) & \text{if } \mu_i < l, \\ q^{4t_i^2 l} w_{2t_i}(q^{-1}) & \text{if } \mu_i = l. \end{cases} $$

In case $m \geqslant n$, we may consider Λ_n^+ as a subset of Λ_m^+. Put $\Lambda_\infty^+ = \bigcup_{m \geqslant 1} \Lambda_m^+$. For $\lambda \in \Lambda_n^+$, we define the transpose $\lambda' \in \Lambda_\infty^+$ of λ by

$$ \lambda_i' = \#\{j; \lambda_j \geqslant i\} \ (i \geqslant 1). \tag{1.5} $$

For $\eta, \tau \in \Lambda_\infty^+$, we put

$$ \langle \eta, \tau \rangle = \sum_{i \geqslant 1} \eta_i \tau_i. \tag{1.6} $$

LEMMA 3. *For $\mu \in \Lambda_{n,l}^+$ and $\xi \in \Lambda_m^+$,*

$$ S_l(\pi^\xi, \pi^\mu) = q^{4lmn - 2m|\hat{\mu}| + 2\langle \xi', \hat{\mu}' \rangle}, $$

where $\hat{\mu} = (l - \mu_n, l - \mu_{n-1}, ..., l - \mu_1)$ $(\in \Lambda_n^+)$.

Proof. By a straightforward calculation, we have

$$S_l(\pi^{\xi}, \pi^{\mu}) = \sum_{T=(t_{ij})} \chi\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \pi^{\mu_i+\xi_j} \times (t_{2j-1,2i-1}t_{2j,2i} - t_{2j,2i-1}t_{2j-1,2i})\right)$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{m} \left| \sum_{a,b\in\mathfrak{o}/\mathfrak{p}^l} \chi(\pi^{\mu_i+\xi_j}ab)\right|^2.$$

From the orthogonality relation of characters we have

$$\sum_{a,b\in\mathfrak{o}/\mathfrak{p}^l} \chi(\pi^{\nu}ab) = q^{l+\nu} \qquad (0 \leqslant \nu \leqslant l).$$

Hence

$$S_l(\pi^{\xi}, \pi^{\mu}) = q^{2lmn + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\min\{l, \mu_i+\xi_j\}}.$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \min\{l, \mu_i+\xi_j\} = lmn + \sum_{\substack{i,j \\ \xi_j \leqslant \hat{\mu}_i}} (\xi_j - \hat{\mu}_i),$$

it suffices to prove the identity

$$\sum_{\substack{i,j \\ \xi_j \leqslant \eta_i}} (\eta_i - \xi_j) = m\,|\eta| - \langle \xi', \eta' \rangle \qquad (\eta \in \Lambda_n^+,\ \xi \in \Lambda_m^+). \tag{1.7}$$

This follows from the formula

$$\sum_{\substack{j=1 \\ \xi_j \leqslant l}}^{m} (l - \xi_j) = \sum_{j=1}^{l} (m - \xi_j').$$

In fact we have

$$\sum_{\substack{i,j \\ \xi_j \leqslant \eta_i}} (\eta_i - \xi_j) = \sum_{i=1}^{n}\sum_{j=1}^{\eta_i} (m - \xi_j') = m\,|\eta| - \sum_{i=1}^{n}\sum_{j=1}^{\eta_i} \xi_j'$$

$$= m\,|\eta| - \sum_{j\geqslant 1} \xi_j' \cdot \#\{i; \eta_i \geqslant j\} = m\,|\eta| - \langle \xi', \eta' \rangle.$$

This proves the lemma.

By Lemmas 1, 2, 3, the calculation of local densities is reduced to the calculation of the Fourier transform $\hat{c}_{\lambda}$ of the function c_{λ} ($\lambda \in \Lambda_n^+$).

2. Zeta Functions on the Space of Alternating Forms

In this section we collect some properties of zeta functions on $V_n(k)$ which are necessary to the calculation of $\hat{c}_\lambda$.

Recall that the determinant of an alternating matrix $x=(x_{ij})$ of size $2n$ is the square of a polynomial $\mathrm{Pf}(x)$ of x_{ij} called the Pfaffian of x, which is given by the formula

$$\mathrm{Pf}(x)=\frac{1}{2^n \cdot n!}\sum_\sigma \mathrm{sgn}(\sigma)\, x_{\sigma(1),\sigma(2)}\cdots x_{\sigma(2n-1),\sigma(2n)},$$

where σ runs through all permutations of $2n$ letters.

For an $x\in V_n(k)$, we denote by $\mathrm{Pf}_i(x)$ $(1\leqslant i\leqslant n)$ the Pfaffian of the upper left $2i$ by $2i$ block of x. For $i=n$, $\mathrm{Pf}_n(x)$ is equal to $\mathrm{Pf}(x)$, the Pfaffian of x. Let $|\omega(x)|_{\mathfrak{p}}$ be the measure on $X_n(k)$ induced by the gauge form

$$\omega(x)=\mathrm{Pf}_n(x)^{-2n+1}\bigwedge_{1\leqslant i<j\leqslant n} dx_{ij} \qquad (x=(x_{ij}),\ x_{ij}=-x_{ji}).$$

For a $\varphi\in\mathscr{S}(V_n(k))$, we define a zeta function by the integral

$$Z(\varphi;s)=\int_{X_n(k)'}\prod_{i=1}^{n}|\mathrm{Pf}_i(x)|_{\mathfrak{p}}^{s_i}\,\varphi(x)\,|\omega(x)|_{\mathfrak{p}} \qquad (s=(s_1,\ldots,s_n)\in\mathbb{C}^n),$$

where

$$X_n(k)'=\{x\in X_n(k);\ \mathrm{Pf}_i(x)\neq 0\ (1\leqslant i\leqslant n)\}.$$

Then, by [HS, Theorem 10], the integral $Z(\varphi;s)$ is absolutely convergent for $\mathrm{Re}(s_1),\ldots,\mathrm{Re}(s_{n-1})>-1$ and $\mathrm{Re}(s_n)>2n-2$. It is known that $Z(\varphi;s)$ represents a rational function of $q^{-s_1},\ldots,q^{-s_n}$ for any $\varphi\in\mathscr{S}(V_n(k))$ (see, e.g. [D]).

Theorem 1. *For any $\varphi\in\mathscr{S}(V_n(k))$, the following functional equation holds*:

$$Z(\hat{\varphi};s)=\gamma(s)\,Z(\varphi^\sigma;s^*),$$

where

$$s^*=(s_{n-1},\ldots,s_1,2n-1-s_1-\cdots-s_n),$$

$$\gamma(s)=q^{-l\sum_{i=1}^n is_i}\prod_{i=1}^{n}\frac{1-q^{s_i+\cdots+s_n-2i+1}}{1-q^{-(s_i+\cdots+s_n-2i+2)}}$$

and

$$\varphi^\sigma(x)=\varphi(\sigma x\sigma) \qquad \text{with} \quad \sigma=\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

The proof is based upon the theory of prehomogeneous vector spaces defined over a $\mathfrak{p}$-adic number field, which is developped by Igusa [I] and then generalized in [S, Section 2] (see also Remark at the end of [S, Section 2]).

Proof. Let $P(k)$ be the subgroup of $G_n(k)$ consisting of lower triangular matrices. Then $P(k)$ acts transitively on $V_n(k)-\bigcup_{i=1}^n \{x\in V_n(k); \mathrm{Pf}_i(x)=0\}$. This means that $(P(k), V_n(k))$ is (the set of k-rational points of) a prehomogeneous vector space defined over k with the singular set $\bigcup_{i=1}^n \{x\in V_n(k); \mathrm{Pf}_i(x)=0\}$. Since Theorem-$k_\mathfrak{p}$ in [S, Section 2] is applicable to $(P(k), V_n(k))$, the quotient $\gamma(s)=Z(\hat{\varphi}; s)/Z(\varphi^\sigma; s^*)$ is independent of $\varphi\in\mathcal{S}(V_n(k))$. Hence the theorem follows immediately from [HS, Theorem 10].

Let $\mathcal{S}(K_n\backslash X_n(k))$ be the subspace of $\mathcal{S}(V_n(k))$ consisting of all K_n-invariant functions with support contained in $X_n(k)$. For example, c_λ is in $\mathcal{S}(K_n\backslash X_n(k))$. For functions φ in $\mathcal{S}(K_n\backslash X_n(k))$ the zeta functions $Z(\varphi; s)$ have been closely investigated in [HS].

Let $z=(z_1, ..., z_n)$ be a variable which is related to $s=(s_1, ..., s_n)$ by the formula

$$\begin{cases} s_1=z_2-z_1-2, \\ \quad \cdots \\ s_{n-1}=z_n-z_{n-1}-2, \\ s_n=(n+1)-z_n-2. \end{cases} \tag{2.1}$$

Denote by $\mathfrak{S}_n$ the symmetric group in n letters and by $\mathbb{C}[q^{\pm z_1}, ..., q^{\pm z_n}]^{\mathfrak{S}_n}$ the ring of symmetric Laurent polynomials in $q^{z_1}, ..., q^{z_n}$. The Hall–Littlewood polynomial $P_\lambda(x; t)$ is defined by

$$\begin{aligned} P_\lambda(x; t)&=P_\lambda(x_1, ..., x_n; t) \\ &=\frac{(1-t)^n}{w_\lambda^{(n)}(t)}\cdot\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n} \\ &\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{x_{\sigma(i)}-tx_{\sigma(j)}}{x_{\sigma(i)}-x_{\sigma(j)}} \end{aligned}$$

for each $\lambda\in\Lambda_n^+$.

For $\varphi \in \mathscr{S}(V_n(k))$, set $\mathscr{Z}(\varphi)(s) = Z(\varphi; s)/Z(c_{\mathbf{0}}; s)$ with $\mathbf{0} = (0, ..., 0) \in \Lambda_n^+$.

THEOREM 2 [HS, Theorems 1, 6]. (i) *For any* $\lambda \in \Lambda_n^+$,

$$Z(c_\lambda; s) = (1 - q^{-1})^n \, q^{(n-1)|\lambda| - 2n(\lambda)} \times \prod_{1 \leqslant i < j \leqslant n} \frac{1 - q^{z_i - z_j - 1}}{1 - q^{z_i - z_j + 1}} \cdot P_\lambda(q^{z_1}, ..., q^{z_n}; q^{-2}).$$

(ii) *For any* $\varphi \in \mathscr{S}(K_n \backslash X_n(k))$ *the function* $\mathscr{Z}(\varphi)$ *is in* $\mathbb{C}[q^{\pm z_1}, ..., q^{\pm z_n}]^{\mathfrak{S}_n}$ *and the mapping*

$$\mathscr{Z}: \mathscr{S}(K_n \backslash X_n(k)) \to \mathbb{C}[q^{\pm z_1}, ..., q^{\pm z_n}]^{\mathfrak{S}_n}$$

is a $\mathbb{C}$*-linear isomorphism.*

Remark. Let $\mathscr{H}(G_n(k), K_n)$ be the Hecke algebra of $G_n(k)$ with respect to K_n. Then we can introduce an $\mathscr{H}(G_n(k), K_n)$-module structure on $\mathscr{S}(K_n \backslash X_n(k))$ by convolution product and on $\mathbb{C}[q^{\pm z_1}, ..., q^{\pm z_n}]^{\mathfrak{S}_n}$ by the Satake isomorphism. Then the mapping $\mathscr{Z}$ defines an isomorphism between these two $\mathscr{H}(G_n(k), K_n)$-modules (for details, see [HS, Section 2]).

Let $\mathscr{S}(K_n \backslash V_n(k))$ be the subspace of $\mathscr{S}(V_n(k))$ consisting of all K_n-invariant functions. The function $\hat{c}_\lambda$ is not in $\mathscr{S}(K_n \backslash X_n(k))$ but in $\mathscr{S}(K_n \backslash V_n(k))$ and our calculation of $\hat{c}_\lambda$ is based on the following lemma.

LEMMA 4. *Let* $\varphi \in \mathscr{S}(K_n \backslash V_n(k))$ *with support in* $V_n(\mathfrak{o})$. *Then* $\mathscr{Z}(\varphi)(s)$ *can be expanded uniquely into an infinite series as*

$$\mathscr{Z}(\varphi)(s) = \sum_{\mu \in \Lambda_n^+} \alpha(\mu) \, P_\mu(q^{z_1}, ..., q^{z_n}; q^{-2}) \tag{2.2}$$

and its coefficients are given by

$$\alpha(\mu) = q^{(n-1)|\mu| - 2n(\mu)} \varphi(\pi^\mu) \qquad (\mu \in \Lambda_n^+).$$

Proof. Since

$$\varphi|_{X_n(k)} = \sum_{\mu \in \Lambda_n^+} \varphi(\pi^\mu) c_\mu,$$

from Theorem 2 (i) we have

$$\mathscr{Z}(\varphi)(s) = \sum_{\mu \in \Lambda_n^+} q^{(n-1)|\mu| - 2n(\mu)} \varphi(\pi^\mu) \, P_\mu(q^{z_1}, ..., q^{z_n}; q^{-2}).$$

Hence it suffices to prove the uniqueness of the expansion. Assume that

$$\sum_{\mu \in \Lambda_n^+} \{\alpha(\mu) - q^{(n-1)|\mu| - 2n(\mu)} \varphi(\pi^\mu)\}\, P_\mu(q^{z_1}, ..., q^{z_n}; q^{-2}) = 0$$

Since $P_\mu(x_1, ..., x_n; q^{-2})$ is homogeneous of degree $|\mu|$, we have

$$\sum_{\substack{\mu \in \Lambda_n^+ \\ |\mu| = d}} \{\alpha(\mu) - q^{(n-1)|\mu| - 2n(\mu)} \varphi(\pi^\mu)\}\, P_\mu(q^{z_1}, ..., q^{z_n}; q^{-2}) = 0$$

for all $d \geqslant 0$. The linear independence of P_μ, which is equivalent to the injectivity of the mapping $\mathscr{S}$ in Theorem 2 (ii), implies that

$$\alpha(\mu) = q^{(n-1)|\mu| - 2n(\mu)} \varphi(\pi^\mu) \qquad (\mu \in \Lambda_n^+).$$

3. Calculation of the Fourier Transform $\hat{c}_\lambda$

To describe the expansion (2.2) for $\mathscr{S}(\hat{c}_\lambda)$, we need some preliminaries. For $\lambda, \mu \in \Lambda_n^+$, we write $\lambda \supset \mu$, if $\lambda_i \geqslant \mu_i$ for all i. We say that $\lambda - \mu$ is a *vertical strip* if $\mu_i' \geqslant \lambda_{i+1}'$ (equivalently, $0 \leqslant \lambda_i - \mu_i \leqslant 1$) for all i (for the definition of μ', see (1.5)).

For $\lambda, \mu, \nu \in \Lambda_n^+$, we can define a polynomial $f_{\mu\nu}^\lambda(t)$ in t by

$$P_\mu(x_1, ..., x_n; t)\, P_\nu(x_1, ..., x_n; t) = \sum_\lambda f_{\mu\nu}^\lambda(t)\, P_\lambda(x_1, ..., x_n; t)$$

(cf. [M, Chap. III, Section 3]).

The Gaussian polynomial $[{n \atop r}](t)$ is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}(t) = \begin{cases} w_n(t)/w_r(t)\, w_{n-r}(t) & \text{if } \ n \geqslant r \geqslant 0, \\ 0 & \text{otherwise} \end{cases}$$

(for the definition of $w_n(t)$, see (1.3)). Then for $0 \leqslant r \leqslant n$,

$$f_{\mu(1^r)}^\lambda(t) = \prod_{i \geqslant 1} \begin{bmatrix} \lambda_i' - \lambda_{i+1}' \\ \lambda_i' - \mu_i' \end{bmatrix}(t), \tag{3.1}$$

where $(1^r) = (\overbrace{1, ..., 1}^{r}, \overbrace{0, ..., 0}^{n-r}) \in \Lambda_n^+$ (cf. [M, Chap. III, (3.2)]).

Put

$$N_\mu^\lambda(t) = t^{n(\mu) - n(\lambda)} \sum_{\nu \in \Lambda_n^+} t^{n(\nu)} f_{\mu\nu}^\lambda(t) \qquad (\lambda, \mu \in \Lambda_n^+).$$

Then by [M, Chap. III, (3.5)], $N_\mu^\lambda(q^{-1})$ is equal to the number of $\mathfrak{o}$-submodules N of $M^\lambda = \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_n}$ such that M^λ/N is isomorphic to $M^\mu = \mathfrak{o}/\mathfrak{p}^{\mu_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\mu_n}$.

LEMMA 5. *For $\lambda, \mu \in \Lambda_n^+$ with $\lambda \supset \mu$,*

$$N_\mu^\lambda(t) = t^{\sum_{i\geqslant 1}(\mu_i'-\lambda_i')\mu_i'} \cdot \prod_{i\geqslant 1} \begin{bmatrix} \lambda_i' - \mu_{i+1}' \\ \mu_i' - \mu_{i+1}' \end{bmatrix}(t).$$

Proof. It suffices to prove the identity for $t = q^{-1}$ by counting the number of $\mathfrak{o}$-submodules N of M^λ such that M^λ/N is isomorphic to M^μ. For a $\lambda \in \Lambda_n^+$, set

$$g^\lambda = \begin{pmatrix} \pi^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{\lambda_n} \end{pmatrix} \in GL(n, k).$$

It is clear that $M^\lambda \simeq \mathfrak{o}^n/g^\lambda\mathfrak{o}^n$. Let N be an $\mathfrak{o}$-submodule of M^λ such that $M^\lambda/N \simeq M^\mu$. Then one can find a matrix A in the double coset $GL(n, \mathfrak{o})\, g^\mu GL(n, \mathfrak{o})$ satisfying $A\mathfrak{o}^n \supset g^\lambda\mathfrak{o}^n$ and $N = A\mathfrak{o}^n/g^\lambda\mathfrak{o}^n$. For $A = ug^\mu v$ with $u, v \in GL(n, \mathfrak{o})$, $A\mathfrak{o}^n$ includes $g^\lambda\mathfrak{o}^n$ if and only if

$$u^{-1} \in g^\mu M(n; \mathfrak{o})(g^\lambda)^{-1} \cap GL(n, \mathfrak{o}).$$

Moreover for $A_1 = u_1 g^\mu v_1$, $A_2 = u_2 g^\mu v_2$ $(u_i, v_i \in GL(n, \mathfrak{o}))$, $A_1\mathfrak{o}^n = A_2\mathfrak{o}^n$ if and only if

$$u_1^{-1}u_2 \in g^\mu GL(n, \mathfrak{o})(g^\mu)^{-1} \cap GL(n, \mathfrak{o}).$$

Therefore we get

$$\begin{aligned} &N_\mu^\lambda(q^{-1}) \\ &\quad = \#[\{g^\mu GL(n; \mathfrak{o})(g^\mu)^{-1} \cap GL(n, \mathfrak{o})\} \backslash \{g^\mu M(n, \mathfrak{o})(g^\lambda)^{-1} \cap GL(n, \mathfrak{o})\}] \\ &\quad = \mathrm{vol}\{g^\mu M(n; \mathfrak{o})(g^\lambda)^{-1} \cap GL(n, \mathfrak{o})\}/\mathrm{vol}\{g^\mu GL(n, \mathfrak{o})(g^\mu)^{-1} \cap GL(n, \mathfrak{o})\}, \end{aligned}$$

where $\mathrm{vol}(X)$ stands for the volume of a subset X of $M(n; k)$ with respect to a Haar measure on $M(n; k)$. Therefore the lemma follows immediately from Lemma 6 below.

LEMMA 6. *We have*

$$\begin{aligned} &\mathrm{vol}\{g^\mu M(n; \mathfrak{o})(g^\lambda)^{-1} \cap GL(n, \mathfrak{o})\} \\ &\quad = q^{\langle \lambda', \mu' \rangle - n|\mu|} \cdot \prod_{j\geqslant 1} \frac{w_{\lambda_j' - \mu_{j+1}'}(q^{-1})}{w_{\lambda_j' - \mu_j'}(q^{-1})}, \end{aligned}$$

where the Haar measure on $M(n; k)$ is so normalized that $\mathrm{vol}(M(n; \mathfrak{o})) = 1$.

Proof. It is obvious that

$$\begin{aligned} &g^{\mu}M(n; \mathfrak{o})(g^{\lambda})^{-1} \cap GL(n, \mathfrak{o}) \\ &\qquad = \{u = (u_{ij}) \in GL(n, \mathfrak{o}); \qquad u_{ij} \in \mathfrak{p}^{\mu_i - \lambda_j}\}. \end{aligned}$$

Let

$$\varphi\colon g^{\mu}M(n; \mathfrak{o})(g^{\lambda})^{-1} \cap GL(n, \mathfrak{o}) \to GL(n, \mathfrak{o}/\mathfrak{p})$$

be the mapping defined by $\varphi(u) = u \pmod{\mathfrak{p}}$. Then the image of φ is given by

$$\mathrm{Im}(\varphi) = \{x = (x_{ij}) \in GL(n, \mathfrak{o}/\mathfrak{p}); \qquad x_{ij} = 0, \qquad \text{if } \ j > L_i(\mu, \lambda)\}$$

and

$$\begin{aligned} \#[\mathrm{Im}(\varphi)] &= \prod_{i=1}^{n} (q^{L_i(\mu,\lambda)} - q^{i-1}) \\ &= q^{\sum_{i=1}^{n} L_i(\mu,\lambda)} \cdot \prod_{i=1}^{n} (1 - q^{-(L_i(\mu,\lambda) - i + 1)}), \end{aligned}$$

where we put $L_i(\mu, \lambda) = \max\{j; \lambda_j \geqslant \mu_i\}$. For $x \in \mathrm{Im}(\varphi)$, the volume of the fibre $\varphi^{-1}(x)$ is independent of x; in fact we have

$$\mathrm{vol}(\varphi^{-1}(x)) = q^{-n^2} \cdot \prod_{\substack{i,j \\ \mu_i > \lambda_j}} q^{-(\mu_i - \lambda_j) + 1}.$$

From the identities

$$\sum_{i=1}^{n} L_i(\mu, \lambda) = n^2 - \sum_{\substack{i,j \\ \mu_i > \lambda_j}} 1,$$

$$\prod_{i=1}^{n} (1 - q^{-(L_i(\mu,\lambda) - i + 1)}) = \prod_{j \geqslant 1} \frac{w_{\lambda'_j - \mu'_{j+1}}(q^{-1})}{w_{\lambda'_j - \mu'_j}(q^{-1})}$$

and (1.7), it follows that

$$\begin{aligned} &\mathrm{vol}\{g^{\mu}M(n; \mathfrak{o})(g^{\lambda})^{-1} \cap GL(n, \mathfrak{o})\} \\ &\quad = \#[\mathrm{Im}(\varphi)] \cdot \mathrm{vol}(\varphi^{-1}(x)) \\ &\quad = q^{\langle \lambda', \mu' \rangle - n|\mu|} \cdot \prod_{j \geqslant 1} \frac{w_{\lambda'_j - \mu'_{j+1}}(q^{-1})}{w_{\lambda'_j - \mu'_j}(q^{-1})}. \end{aligned}$$

LEMMA 7. *We have*

$$\begin{aligned}
P_\lambda(x_1, ..., x_n; t) \cdot \prod_{i=1}^{n} \frac{1+x_i y}{1-x_i}
&= \sum_{\mu \in \Lambda_n^+} P_\mu(x_1, ..., x_n; t) \cdot \left\{ \sum_{\nu} y^{|\mu| - |\nu|} t^{n(\nu) - n(\lambda) + \sum_{i \geqslant 1} (\lambda_i' - \nu_i') \lambda_i'} \right. \\
&\quad \left. \times \prod_{i \geqslant 1} \left(\begin{bmatrix} \mu_i' - \mu_{i+1}' \\ \mu_i' - \nu_i' \end{bmatrix} (t) \begin{bmatrix} \nu_i' - \lambda_{i+1}' \\ \lambda_i' - \lambda_{i+1}' \end{bmatrix} (t) \right) \right\},
\end{aligned}$$

where the second summation is taken over all $\nu \in \Lambda_n^+$ *such that* $\mu - \nu$ *is a vertical strip and* $\lambda \subset \nu \subset \mu$.

Proof. Since

$$\prod_{i=1}^{n} \frac{1}{1-x_i} = \sum_{\nu \in \Lambda_n^+} t^{n(\nu)} P_\nu(x_1, ..., x_n; t),$$

we have by Lemma 5

$$\begin{aligned}
P_\lambda(x_1, ..., x_n; t) \prod_{i=1}^{n} \frac{1}{1-x_i}
&= \sum_{\nu \in \Lambda_n^+} P_\nu(x_1, ..., x_n; t) t^{n(\nu) - n(\lambda)} N_\lambda^\nu(t) \\
&= \sum_{\substack{\nu \in \Lambda_n^+ \\ \lambda \subset \nu}} P_\nu(x_1, ..., x_n; t) \\
&\quad \times t^{n(\nu) - n(\lambda) + \sum_{i \geqslant 1} (\lambda_i' - \nu_i') \lambda_i'} \cdot \prod_{i \geqslant 1} \begin{bmatrix} \nu_i' - \lambda_{i+1}' \\ \lambda_i' - \lambda_{i+1}' \end{bmatrix} (t).
\end{aligned}$$

Recall that $P_{(1^r)}(x_1, ..., x_n; t)$ is the rth elementary symmetric polynomial [M, Chap. III, (2.8)] and that

$$\prod_{i=1}^{n} (1 + x_i y) = \sum_{r=0}^{n} y^r P_{(1^r)}(x_1, ..., x_n; t).$$

Therefore the formula for

$$P_\lambda(x_1, ..., x_n; t) \cdot \prod_{i=1}^{n} \frac{1+x_i y}{1-x_i}$$

in the lemma is an immediate consequence of the identity (3.1).

THEOREM 3. *For* $\lambda, \mu \in \Lambda_n^+$,

$$\hat{c}_\lambda(\pi^\mu) = (-1)^n q^{-n^2(2l-1)+(2n-1)|\lambda|-4n(\lambda)+|\mu|+2n(\mu)}$$
$$\times \sum_{\substack{\nu \in \Lambda_n^+ \\ \nu \subset \mu}} (-1)^{|\mu|-|\nu|} q^{e(\lambda,\nu)}$$
$$\times \prod_{i \geqslant 1} \left\{ \begin{bmatrix} \mu_i' - \mu_{i+1}' \\ \mu_i' - \nu_i' \end{bmatrix} (q^{-2}) \begin{bmatrix} \tilde{\lambda}_{l-i}' - (n-\nu_i') \\ \tilde{\lambda}_{l-i}' - \tilde{\lambda}_{l-i+1}' \end{bmatrix} (q^{-2}) \right\},$$

where $\tilde{\lambda} = (\lambda_1 + 1, ..., \lambda_n + 1)$ *and*

$$e(\lambda, \nu) = (2n-1)\,|\nu| - 2n(\nu) - 2 \sum_{i \geqslant 1} \nu_i' \tilde{\lambda}_{l-i+1}'$$

with the understanding that $\tilde{\lambda}_j' = n$ *for* $j \leqslant 0$.

Proof. For $s = (s_1, ..., s_n)$ and $z = (z_1, ..., z_n) \in \mathbb{C}^n$, put

$$s^* = (s_{n-1}, ..., s_1, 2n-1-(s_1 + \cdots + s_n))$$

and

$$z^* = (-z_n - 2n + 1, \ -z_{n-1} - 2n + 1, ..., \ -z_1 - 2n + 1).$$

Since $P_{\mathbf{0}}(q^{z_1}, ..., q^{z_n}; q^{-2}) = 1$ for $\mathbf{0} = (0, ..., 0) \in \Lambda_n^+$, it follows from Theorem 2 (i) that $Z(c_{\mathbf{0}}; s^*) = Z(c_{\mathbf{0}}; s)$. Therefore by Theorem 1 and Theorem 2 (i), we have

$$\mathscr{L}(\hat{c}_\lambda)(s) = (-1)^n q^{-2n(\lambda)-n|\lambda|-n^2} \prod_{i=1}^n \frac{1-q^{z_i+n}}{1-q^{z_i+n-1}}$$
$$\times q^{(l-1)\sum_{i=1}^n z_i} P_\lambda(q^{-z_1}, ..., q^{-z_n}; q^{-2})$$
$$= (-1)^n q^{-2n(\lambda)-n|\lambda|-n^2}$$
$$\times \prod_{i=1}^n \frac{1-q^{z_i+n}}{1-q^{z_i+n-1}} P_{(l)-\tilde{\lambda}}(q^{z_1}, ..., q^{z_n}; q^{-2}),$$

where we put $(l) - \tilde{\lambda} = (l-1-\lambda_n, ..., l-1-\lambda_1)$. Substituting $x_1, ..., x_n, t, y$ in the formula in Lemma 7 by $q^{z_1+n-1}, ..., q^{z_n+n-1}, q^{-2}, -q$, respectively, we obtain

$$\mathscr{L}(\hat{c}_\lambda)(s) = (-1)^n q^{-2n(\lambda)-|\lambda|-ln(n-1)-n} \sum_{\mu \in \Lambda_n^+} P_\mu(q^{z_1}, ..., q^{z_n})$$
$$\times q^{(n-1)|\mu|} \sum_{\substack{\nu \in \Lambda_n^+ \\ \nu \subset \mu}} (-1)^{|\mu|-|\nu|} q^{d(\lambda,\nu)+|\mu|}$$
$$\times \prod_{i \geqslant 1} \left(\begin{bmatrix} \mu_i' - \mu_{i+1}' \\ \mu_i' - \nu_i' \end{bmatrix} (q^{-2}) \begin{bmatrix} \tilde{\lambda}_{l-i}' - (n-\nu_i') \\ \tilde{\lambda}_{l-i}' - \tilde{\lambda}_{l-i+1}' \end{bmatrix} (q^{-2}) \right),$$

where

$$d(\lambda, \nu) = -|\nu| - 2n(\nu) + 2n((l) - \tilde{\lambda})$$
$$+ 2 \sum_{i \geqslant 1} (\nu'_i + \tilde{\lambda}'_{l-i+1} - n)(n - \tilde{\lambda}'_{l-i+1}).$$

Therefore by Lemma 4 and the formula

$$\langle \mu', \mu' \rangle = 2n(\mu) + |\mu| \qquad (\mu \in \Lambda_n^+)$$

we get the theorem.

Now we are in a position to prove our main theorem.

THEOREM 4. *For $\lambda \in \Lambda_n^+$ and $\xi \in \Lambda_m^+$,*

$$\mu(\pi^\lambda, \pi^\xi) = \sum_{\substack{\mu \in \Lambda_n^+ \\ \mu \subset \tilde{\lambda}}} (-1)^{|\mu|} q^{2(n-m-1)|\mu| - 2n(\mu) + 2\langle \xi', \mu' \rangle}$$
$$\times \prod_{\substack{i \geqslant 1 \\ \mu'_i \neq \mu'_{i+1}}} \left\{ \sum_{l = \mu'_{i+1}}^{\min\{\tilde{\lambda}'_{i+1}, \mu'_i\}} (-q)^{l(2\tilde{\lambda}'_{i+1} - l)} \right.$$
$$\left. \times \begin{bmatrix} \tilde{\lambda}'_{i+1} - \mu'_{i+1} \\ \tilde{\lambda}'_{i+1} - l \end{bmatrix} (q^{-2}) \begin{bmatrix} \tilde{\lambda}'_i - l \\ \tilde{\lambda}'_i - \mu'_i \end{bmatrix} (q^{-2}) \right\},$$

where $\tilde{\lambda} = (\lambda_1 + 1, \ldots, \lambda_n + 1)$.

Proof. By Lemmas 1, 2, 3, and Theorem 3, we have

$$\mu(\pi^\lambda, \pi^\xi) = (-1)^n q^{n^2 - ln} w_\lambda^{(n)}(q^{-2})$$
$$\times \sum_{\mu \in \Lambda_{n,l}^+} (-1)^{|\mu|} q^{-2m|\hat{\mu}| - 2n(\mu) + 2\langle \xi', \hat{\mu}' \rangle}$$
$$\times \frac{1}{w_\mu^{(n)}(q^{-2})} \cdot \frac{w_{m_l(\mu)}(q^{-2})}{w_{2m_l(\mu)}(q^{-1})}$$
$$\sum_{\substack{\nu \\ (\tilde{\lambda})^\wedge \subset \nu \subset \mu}} (-q)^{-2n(\nu) + (2n-1)|\nu| + 2\langle \tilde{\lambda}', \hat{\nu}' \rangle}$$
$$\times \prod_{i \geqslant 1} \begin{bmatrix} \hat{\mu}'_{i-1} - \hat{\mu}'_i \\ \hat{\nu}'_i - \hat{\mu}'_i \end{bmatrix} (q^{-2}) \begin{bmatrix} \tilde{\lambda}'_i - \hat{\nu}'_{i+1} \\ \tilde{\lambda}'_i - \tilde{\lambda}'_{i+1} \end{bmatrix} (q^{-2})$$

(for the definition of $\hat{\mu}$, see Lemma 3). Note that the mapping $\mu \mapsto \hat{\mu}$ defines a bijection of $\Lambda_{n,l}^{+}$ onto itself and the following identities hold:

$$m_l(\mu) = m_0(\hat{\mu}), \qquad |\hat{\mu}| + |\mu| = ln,$$
$$n(\mu) = \frac{ln(n-1)}{2} - (n-1)\,|\hat{\mu}| + n(\hat{\mu}).$$

Hence we have

$$\begin{aligned}
\mu(\pi^{\lambda}, \pi^{\xi}) &= (-1)^n q^{-n^2} w_{\lambda}^{(n)}(q^{-2}) \\
&\times \sum_{\mu \in \Lambda_{n,l}^{+}} (-1)^{|\mu|} q^{2(n-m-1)|\mu| - 2n(\mu) + 2\langle \xi', \mu' \rangle} \\
&\times \frac{1}{w_{\mu}^{(n)}(q^{-2})} \cdot \frac{w_{m_0(\mu)}(q^{-2})}{w_{2m_0(\mu)}(q^{-1})} \\
&\times \sum_{\substack{\nu \\ \bar{\lambda} \supset \nu \supset \mu}} (-q)^{-2n(\nu) - |\nu| + 2\langle \bar{\lambda}', \nu' \rangle} \\
&\times \prod_{i \geqslant 1} \begin{bmatrix} \mu'_{i-1} - \mu'_i \\ \nu'_i - \mu'_i \end{bmatrix} (q^{-2}) \begin{bmatrix} \bar{\lambda}'_i - \nu'_{i+1} \\ \bar{\lambda}'_i - \bar{\lambda}'_{i+1} \end{bmatrix} (q^{-2}).
\end{aligned}$$

The summation with respect to ν in the right hand side of this identity can be written as

$$\begin{aligned}
\prod_{i \geqslant 1} & \sum_{k = \mu'_i}^{\min\{\bar{\lambda}'_i, \mu'_{i-1}\}} (-q)^{-k^2 + 2\bar{\lambda}'_i k} \\
& \times \begin{bmatrix} \mu'_{i-1} - \mu'_i \\ k - \mu'_i \end{bmatrix} (q^{-2}) \begin{bmatrix} \bar{\lambda}'_{i-1} - k \\ \bar{\lambda}'_{i-1} - \bar{\lambda}'_i \end{bmatrix} (q^{-2}).
\end{aligned}$$

Using the formula

$$\begin{aligned}
& \begin{bmatrix} \mu'_{i-1} - \mu'_i \\ k - \mu'_i \end{bmatrix} (t) \begin{bmatrix} \bar{\lambda}'_{i-1} - k \\ \bar{\lambda}'_{i-1} - \bar{\lambda}'_i \end{bmatrix} (t) \\
&\quad = \begin{bmatrix} \bar{\lambda}'_i - \mu'_i \\ \bar{\lambda}'_i - k \end{bmatrix} (t) \begin{bmatrix} \bar{\lambda}'_{i-1} - k \\ \bar{\lambda}'_{i-1} - \mu'_{i-1} \end{bmatrix} (t) \\
&\quad \frac{w_{m_{i-1}(\mu)}(t)}{w_{m_{i-1}(\bar{\lambda})}(t)} \cdot \frac{w_{\bar{\lambda}'_{i-1} - \mu'_{i-1}}(t)}{w_{\bar{\lambda}'_i - \mu'_i}(t)},
\end{aligned}$$

we can see that the summation with respect to v is equal to

$$\frac{w_{\mu}^{(n)}(q^{-2})}{w_{\lambda}^{(n)}(q^{-2})} \cdot \prod_{i \geqslant 1} \left\{ \sum_{k=\mu_i'}^{\min\{\bar{\lambda}_i', \mu_{i-1}'\}} (-q)^{-k^2+2\bar{\lambda}_i' k} \right.$$
$$\left. \times \begin{bmatrix} \bar{\lambda}_i' - \mu_i' \\ \bar{\lambda}_i' - k \end{bmatrix} (q^{-2}) \begin{bmatrix} \bar{\lambda}_{i-1}' - k \\ \bar{\lambda}_{i-1}' - \mu_{i-1}' \end{bmatrix} (q^{-2}) \right\}.$$

The summation with respect to k is equal to 1 if $\mu_i' = \mu_{i-1}'$, and to

$$(-1)^n q^{n^2} w_{2m_0(\mu)}(q^{-1}) / w_{m_0(\mu)}(q^{-2})$$

if $i = 1$. This can easily be seen by using the formula

$$\sum_{r=0}^{n} t^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} (t) X^r = \prod_{r=0}^{n-1} (1 + t^i X)$$

(cf. [M, Chap. I, Section 2, Example 2]). Thus we get

$$\mu(\pi^{\lambda}, \pi^{\xi}) = \sum_{\mu \subset \bar{\lambda}} (-1)^{|\mu|} q^{2(n-m-1)|\mu| - 2n(\mu) + 2\langle \xi', \mu' \rangle}$$
$$\times \prod_{\substack{i \geqslant 1 \\ \mu_i' \neq \mu_{i+1}'}} \left\{ \sum_{l=\mu_{i+1}'}^{\min\{\bar{\lambda}_{i+1}', \mu_i'\}} (-q)^{l(2\bar{\lambda}_{i+1}' - l)} \right.$$
$$\left. \times \begin{bmatrix} \bar{\lambda}_{i+1}' - \mu_{i+1}' \\ \bar{\lambda}_{i+1}' - l \end{bmatrix} (q^{-2}) \begin{bmatrix} \bar{\lambda}_i' - l \\ \bar{\lambda}_i' - \mu_i' \end{bmatrix} (q^{-2}) \right\}.$$

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